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THE BRAUER GROUP OF A CURVE OVER A STRICTLY LOCAL DISCRETE VALUATION RING

BY

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ABSTRACT

Let K be the field of fractions of a curve over R where R is the henselization of a regular local ring on an algebraic curve over a field which is algebraically closed and has characteristic 0. Then K has the exponent = degree property for division algebras. In fact every central finite dimensional K-division algebra with exponent n is a cyclic algebra of degree n.

In this paper we continue to investigate the structure of division algebras D finite dimensional over their center K. The motivating problem is to classify those fields K that have the **exponent = degree** property for division algebras. We say that K has the **exponent = degree** property if for any central K-division algebra D the exponent of the class $[D]$ in the Brauer group $\mathrm{B}(K)$ is equal to the degree $\sqrt{(D\colon K)}$ of the division algebra. Throughout this paper k is an algebraically closed field of characteristic 0.

Example 1: Some fields that are known to have the exponent = degree property are listed below.

(1) A global field (an algebraic number field or a function field finitely generated of transcendence degree 1 over a finite field). This is classical.
(2) The quotient field of either (a) the henselization $\mathcal{O}^h_{p,X}$ or (b) the completion $\hat{\mathcal{O}}_{p,X}$ at a closed point p on a normal surface X over k [1] or [6].

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(3) The quotient field of a ring obtained by (a) henselizing or (b) completing an affine surface over k along an integral curve [4].

In fact in each of these three examples, each division algebra D is split by a cyclic extension $K(\alpha^{1/n})$ for some $\alpha \in K$ and $n = \text{exponent}(D)$.

The purpose of this paper is to add to the list of Example 1 another class of fields satisfying the exponent = degree property.

Let $\mathcal{O}_{p,X}$ be the local ring at a regular point p on an algebraic curve X over the field k. Then $\mathcal{O}_{p,X}$ is a local principal ideal domain, hence a discrete valuation ring. The residue field of $\mathcal{O}_{p,X}$ is k. Let $R = \mathcal{O}^h_{p,X}$ be the henselization of $\mathcal{O}_{p,X}$. Consider an affine algebraic curve C over R. Following [9], C is an affine scheme together with a structure morphism $\pi\colon C \longrightarrow \operatorname{Spec} R$ such that π is flat and of finite type, the fibers of π are algebraic curves, and C is connected. Then π has 2 fibers. The closed fiber $\pi\colon C_0 \longrightarrow x_0$ over the closed point x_0 of $\operatorname{Spec} R$ is an algebraic curve over k. The open fiber $\pi\colon C_\eta \longrightarrow \eta$ over the open point of $\operatorname{Spec} R$ is an algebraic curve over the quotient field of R. Assume that C_η is integral, with $K = K(C_\eta)$ the field of fractions. Our main result is that K has the exponent = degree property for division algebras.

THEOREM 2: *Let R, C and K be as above and let D be a central finite dimensional K-division algebra with* exponent$(D) = n$. *Then D is a cyclic algebra of degree n.*

Proof: The proof is in the flavor of those used by [6] and [4].

Since C is $\operatorname{Spec} S$ for an algebra S of finite type over R, we can assume C is a closed subscheme of affine space $\mathbb{A}^m_R$ over R. Without changing K we can replace C with a projective completion over R. If necessary, we can also desingularize C. Therefore assume that $\pi\colon C \longrightarrow \operatorname{Spec} R$ is proper, that the open fiber is a nonsingular integral curve C_η over the quotient field of R. By Embedded Resolution of Curves in Surfaces [8, p. 391], we can assume that the closed fiber $(C_0)_{red}$ is a divisor over k with normal crossings. That is, write the reduced closed fiber $(C_0)_{red}$ as a union $C_1 \cup C_2 \cup \cdots \cup C_s$ of irreducible curves. By the normal crossing hypothesis we assume each component C_j is a nonsingular curve and that $(C_0)_{red}$ has at most ordinary double points as singularities.

Let L/K be a finite extension of fields and $Y \longrightarrow C$ the integral closure of C in L. Let $f\colon Y' \longrightarrow Y$ be any desingularization of Y. That is, Y' is nonsingular and f is a proper birational morphism. There is a complex

$$(1)\qquad \begin{aligned} 0 \longrightarrow \mathrm{B}(Y') \longrightarrow \mathrm{B}(L) \xrightarrow{a} \bigoplus_{\Delta} H^1(K(\Delta), \mathbb{Q}/\mathbb{Z}) \\ \xrightarrow{r} \bigoplus_{P} \mu(-1) \xrightarrow{s} H^4(Y', \mu) \longrightarrow 0 \end{aligned}$$

which is exact except possibly at the term $\bigoplus H^1(K(\Delta), \mathbb{Q}/\mathbb{Z})$. The first summation is over all irreducible curves $\Delta \subseteq Y'$, the second over all closed points $P \in Y'$. This follows by combining sequences (3.1) and (3.2) of [2]. If $H^3(Y', \mu) = 0$, (1) is exact. The first two groups in (1) are the Brauer groups respectively, of Y' and L. The map a "measures the ramification" of a division algebra Λ over L. The ramification divisor of Λ is the set of divisors Δ where $a[\Lambda]$ is nontrivial. The group $H^1(K(\Delta), \mathbb{Q}/\mathbb{Z})$ classifies the cyclic Galois extensions of $K(\Delta)$, the function field of Δ. The map r measures the ramification of cyclic extensions of $K(\Delta)$. Here $\mu(-1) = \bigcup_n \mathrm{Hom}(\mu_n, \mathbb{Q}/\mathbb{Z})$. Let D be a central K-division algebra and $D_L = D \otimes L$, the restriction of D to L. We say that L splits the ramification of D on C if there exists a desingularization $f \colon Y' \longrightarrow Y$ such that the class of D_L in the Brauer group $\mathrm{B}(L)$ is in the image of the Brauer group $\mathrm{B}(Y')$ of Y'.

We proceed as in the proof of [4, Cor. 5]. Since R is a direct limit of étale neighborhoods of (X, p), C is of finite type over R and D is a finite K-algebra, we can find an étale neighborhood (U, p) of (X, p), and a nonsingular algebraic surface C_1 satisfying the following:

(1) There is a proper morphism $C_1 \longrightarrow U$.

(2) $C = C_1 \times_U \operatorname{Spec} R$.

(3) If K_1 is the function field of C_1, then there is a central simple algebra D_1 over K_1 such that $D = D_1 \otimes_{K_1} K$.

It was shown in the text immediately preceding Theorem 1.6 of [6] and again in [4, Prop. 3] that there exists a surface C_2 and a proper birational morphism $C_2 \longrightarrow C_1$ and a cyclic field extension L_1/K_1 of degree n such that L_1 splits the ramification of D_1 on C_2. Furthermore, if Y_1 is the integral closure of C_2 in L_1, then Y_1 has only rational singularities. Let L denote the field KL_1. Then L/K is cyclic of degree n. Set $C' = C_2 \times_U \operatorname{Spec} R$. Let Y denote the integral closure of C' in L. By the construction of C_2, Y has at most rational singularities and $D \otimes L$ is unramified at each prime divisor on Y. There is a desingularization $Y' \longrightarrow Y$ and $D \otimes L$ is unramified on Y'. That is, $D \otimes L$ represents a class in the image of the Brauer group of Y'. It therefore suffices to show that Y' has trivial Brauer group. But $Y' \longrightarrow R$ satisfies the hypothesis of Theorem 3 (which is stated and proved below), so $D \otimes L$ is split. It follows that D is a cyclic algebra of degree n. ∎

THEOREM 3: *Let R, C and K be as in Theorem 2. Assume moreover that $\pi: C \longrightarrow \operatorname{Spec} R$ is proper, C is regular, that the fibers of π are one dimensional, and that the closed fiber of π is a curve over k with normal crossings. Then $H^q(C, \mu) = 0$ for all $q \geq 3$ and $H^q(C, \mathbb{G}_m) = 0$ for all $q \geq 2$.*

Proof: Fix an integer $n \geq 2$. By proper base change $H^q(C, \mu_n) \cong H^q(C_0, \mu_n)$ for all $q \geq 1$, where C_0 is the closed fiber of π (i.e. $C_0 = C \times_R x_0$ where x_0 is the closed point of $\operatorname{Spec} R$). Since C_0 is a curve over $x_0 = \operatorname{Spec} k$, $H^q(C_0, \mu_n) = 0$ for $q \geq 3$. Taking the direct limit over all n gives $H^q(C_0, \mu) = 0$ for $q \geq 3$. The sequence of sheaves for the étale topology on C

$$(2) \qquad 1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{x \longrightarrow x^n} \mathbb{G}_m \longrightarrow 1$$

is exact by Kummer theory. The associated long exact sequence

$$(3) \qquad \cdots \longrightarrow H^q(C, \mu_n) \longrightarrow H^q(C, \mathbb{G}_m) \xrightarrow{n} H^q(C, \mathbb{G}_m) \longrightarrow \cdots$$

shows that multiplication by n is an isomorphism on $H^q(C, \mathbb{G}_m)$ for $q \geq 3$. Since C is regular, by [7, II, p. 71] $H^q(C, \mathbb{G}_m)$ is a torsion group for all $q \geq 2$. Therefore $H^q(C, \mathbb{G}_m) = 0$ for all $q \geq 3$. Now we check that the Brauer group of C, $\mathrm{B}(C) = H^2(C, \mathbb{G}_m)$, is trivial. We use the Kummer sequence (3) for $q = 2$

$$(4) \qquad 0 \longrightarrow \frac{\operatorname{Pic} C}{n \operatorname{Pic} C} \longrightarrow H^2(C, \mu_n) \longrightarrow {}_n\mathrm{B}(C) \longrightarrow 0$$

together with the fact that $H^2(C, \mu_n) \cong H^2(C_0, \mu_n)$. We assume C_0 is reduced, since $H^2(C_0, \mu_n) \cong H^2((C_0)_{\mathrm{red}}, \mu_n)$. Write $C_0 = C_1 \cup C_2 \cup \cdots \cup C_s$ as a union of nonsingular irreducible curves. We assume each component C_j is a nonsingular curve by the normal crossing hypothesis. By the Kummer sequence (4), the known description of $\operatorname{Pic} C_j$ and the fact that $\mathrm{B}(C_j) = 0$ (see for example [10, pp. 175–176]), it follows that $H^2(C_j, \mu_n) \cong \mathbb{Z}/n$ is generated by the class of any prime divisor on C_j. Now $H^2(C_0, \mu_n) \cong \coprod_{j=1}^{s} H^2(C_j, \mu_n)$ by Lemma 4 below. It suffices to show that for each $j = 1, \ldots, s$, there exists a divisor D_j on C such that $D_j \cap C_j$ is a prime divisor on C_j and $D \cap C_i = \emptyset$ if $i \neq j$. Let P_0 be a prime divisor on C_j not in the singular locus of C_0 and pick any prime divisor D_j on C such that the intersection multiplicity of D_j and C_j at the closed point P_0 is 1. This is possible since C and C_j are both regular at P_0. The problem that one must worry about is the possibility that D_j intersects C_0 at some other point. We check that this cannot happen since D_j is integral and R is henselian. Now $\iota\colon D_j \hookrightarrow C$ is a closed immersion, hence is proper. Furthermore D_j does not contain any component C_i of C_0. Also D_j is closed, so D_j does not contain C_η. Consider the composite $f = \pi \circ \iota\colon D_j \longrightarrow \operatorname{Spec} R$. Since the fibers of $\bar{\pi}$ are of dimension 1, $f^{-1}(x)$ is finite for each x in $\operatorname{Spec} R$. So f is quasi-finite. Since f is a composite of proper morphisms, f is proper. But a proper quasi-finite morphism is finite [10, p. 6] so D_j is finite over R. Since R is henselian, any connected component of $D_j \times_R x_0$ gives rise to a connected component of D_j. But D_j is integral, hence $D_j \times_R x_0 = P_0$. So ${}_n\mathrm{B}(C) = 0$ for each $n > 1$. ∎

LEMMA 4: *Let C be an algebraic curve over k embedded with normal crossings on a surface S. If the irreducible components of C are $C_1, \ldots, C_s$, then $H^2(C, \mu_n) \cong \coprod_{i=1}^{n} H^2(C_i, \mu_n)$.*

Proof: Let $D = C_1 \coprod \cdots \coprod C_s$ denote the disjoint union of the curves $C_1, \ldots, C_s$. There is an obvious finite projection $\pi\colon D \longrightarrow C$. Let σC denote the singular locus of C and $\sigma D = \pi^{-1}(\sigma C)$ those points on D lying over σC. Since C has only nodal singularities, the map $\pi\colon \sigma D \longrightarrow \sigma C$ is 2-to-1. Let P be an element of σC and consider the cohomology with supports in P, $H^2_P(C, \mu_n)$. The strictly local ring $\mathcal{O}^h_{C,P}$ is henselian with algebraically closed residue field k. Let U denote $\operatorname{Spec} \mathcal{O}^h_{C,P}$ and let P^h denote the closed point of U. By excision [10, Cor. 1.28, p. 93] $H^2_P(C, \mu_n) \cong H^2_{P^h}(U, \mu_n)$. The long exact sequence for $P^h \subseteq U$ is

$$\begin{aligned} \cdots \longrightarrow H^1(U, \mu_n) \longrightarrow H^1(U - P^h, \mu_n) \longrightarrow H^2_{P^h}(U, \mu_n) \\ \longrightarrow H^2(U, \mu_n) \longrightarrow \cdots. \end{aligned} \tag{5}$$

The curve U consists of 2 nonsingular henselian curves U_1, U_2 crossing at the closed point P^h. Each curve U_i is the prime spectrum of a henselian discrete valuation ring with residue field k. So $H^i(U, \mu_n) = 0$ for $i > 0$ and

$$H^1(U - P^h, \mu_n) = H^1(U_1 - P^h, \mu_n) \oplus H^1(U_2 - P^h, \mu_n) \cong \mathbb{Z}/n \oplus \mathbb{Z}/n.$$

So equation (5) and excision show that $H^2_P(C, \mu_n) \cong (\mathbb{Z}/n)^{(2)}$. If Q is an element of σD, then the argument above also shows that $H^2_Q(D, \mu_n) \cong \mathbb{Z}/n$. Since

σC decomposes into a finite number of points P, it follows that $H^2_{\sigma C}(C, \mu_n)$ decomposes into the direct sum $\coprod_{P \in \sigma C} H^2_P(C, \mu_n)$ and similarly for σD. The long exact sequences of cohomology with supports in σC and σD combined with the maps induced by π yield the commutative diagram below.

(6)

$$\begin{array}{ccccccc} H^1(C - \sigma C, \mu_n) & \longrightarrow & H^2_{\sigma C}(C, \mu_n) & \longrightarrow & H^2(C, \mu_n) & \longrightarrow & H^2(C - \sigma C, \mu_n) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta \\ H^1(D - \sigma D, \mu_n) & \longrightarrow & H^2_{\sigma D}(D, \mu_n) & \longrightarrow & H^2(D, \mu_n) & \longrightarrow & H^2(D - \sigma D, \mu_n) \end{array}$$

Because $\pi\colon C - \sigma C \xrightarrow{\cong} D - \sigma D$, α and δ are isomorphisms. The map β is an isomorphism by the above computations. Therefore γ is an isomorphism. Because D is a disjoint union, the lemma follows. ∎

COROLLARY 5: *Let R, C and K be as in Theorem 3. The sequence*

$$0 \longrightarrow \mathrm{B}(K) \xrightarrow{a} \bigoplus_{\Delta} H^1(K(\Delta), \mathbb{Q}/\mathbb{Z}) \xrightarrow{r} \bigoplus_{P} \mu(-1) \longrightarrow 0$$

is exact where the first summation is over all irreducible curves $\Delta \subseteq C$, the second over all closed points $P \in C$.

Proof: Follows immediately from (1) and Theorem 3. ∎

For any discrete valuation ring R with perfect residue field k and field of fractions K, for each $q \geq 0$ the natural map $H^q(R, \mathbb{G}_m) \longrightarrow H^q(k, \mathbb{G}_m)$ is an isomorphism [7, III, p. 93]. For $q = 2$ this is the theorem of Azumaya. There is a split-exact sequence

$$0 \longrightarrow H^q(k, \mathbb{G}_m) \longrightarrow H^q(K, \mathbb{G}_m) \longrightarrow H^{q-1}(k, \mathbb{G}_m) \longrightarrow 0$$

for each $q \geq 2$ [7, III, p. 93 and p. 188]. In particular, for $q = 2$, it follows that the Brauer group of K decomposes into a direct sum of $\mathrm{B}(k)$ and $H^1(k, \mathbb{Q}/\mathbb{Z})$. The group $H^1(k, \mathbb{Q}/\mathbb{Z})$ parametrizes the unramified cyclic Galois extensions of K. Various other results along these same lines are derived in [3] and [11]. We arrive at similar results for the Brauer group of a curve over K.

THEOREM 6: *Let R be a strictly local ring which is the henselization of a local ring $\mathcal{O}_{p,X}$ at a closed point p on a smooth curve X over k. Let $\pi\colon C \longrightarrow \operatorname{Spec} R$ be proper and smooth of relative dimension 1. Let η be the generic point of* Spec R

and $C_\eta = C \times_R \eta$. Let x_0 be the closed point of $\operatorname{Spec} R$ *and $C_0 = C \times_R x_0$. Then the Brauer group $\mathrm{B}(C_\eta)$ is isomorphic to $H^1(C_0, \mathbb{Q}/\mathbb{Z})$. Every Azumaya algebra over C_η of exponent n is split by a cyclic Galois extension of degree n which descends to an unramified extension of C.*

Proof: Since $\operatorname{Spec} R = \{x_0\} \cup \{\eta\}$ with $\{x_0\}$ closed and $\{\eta\}$ open, we have $C = C_0 \cup C_\eta$ with C_0 closed and C_η open. The long exact sequence of cohomology with supports in C_0 and coefficients in $\mathbb{G}_m$ is

$$\begin{array}{l} H^0(C,\mathbb{G}_m) \longrightarrow H^0(C_\eta,\mathbb{G}_m) \longrightarrow H^1_{C_0}(C,\mathbb{G}_m) \longrightarrow \\ H^1(C,\mathbb{G}_m) \longrightarrow H^1(C_\eta,\mathbb{G}_m) \longrightarrow H^2_{C_0}(C,\mathbb{G}_m) \longrightarrow \\ H^2(C,\mathbb{G}_m) \longrightarrow H^2(C_\eta,\mathbb{G}_m) \longrightarrow H^3_{C_0}(C,\mathbb{G}_m) \longrightarrow \cdots \end{array} \tag{7}$$

If t is a local parameter for R, then

$$\frac{H^0(C_\eta,\mathbb{G}_m)}{H^0(C,\mathbb{G}_m)} = \langle t \rangle$$

and t vanishes with order 1 along C_0. Since C_0 is a principal divisor, $\operatorname{Pic} C \cong \operatorname{Pic} C_\eta$. From Kummer theory the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(C_\eta,\mathbb{G}_m)\otimes\mathbb{Z}/n & \longrightarrow & H^1(C_\eta,\mu_n) & \longrightarrow & {}_n\operatorname{Pic} C_\eta & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow\cong & & \\ 0 & \longrightarrow & H^0(C,\mathbb{G}_m)\otimes\mathbb{Z}/n & \longrightarrow & H^1(C,\mu_n) & \longrightarrow & {}_n\operatorname{Pic} C & \longrightarrow & 0 \end{array} \tag{8}$$

commutes and has exact rows. Since $H^0(C_\eta,\mathbb{G}_m)\otimes\mathbb{Z}/n \cong \langle t\rangle/\langle t^n\rangle$ and $H^0(C,\mathbb{G}_m)\otimes\mathbb{Z}/n = 0$, we see from (8) that $H^1(C_\eta,\mu_n) \cong \langle t\rangle/\langle t^n\rangle \times H^1(C,\mu_n)$ and by proper base change $H^1(C,\mu_n) \cong H^1(C_0,\mu_n)$. From Theorem 3 we have $H^q(C,\mathbb{G}_m) = 0$ for all $q \geq 2$. From (7) it follows that $H^q(C_\eta,\mathbb{G}_m) \cong H^{q+1}_{C_0}(C,\mathbb{G}_m)$ for all $q \geq 2$. Since $\operatorname{Spec} R$ is a direct limit of étale neighborhoods $U \hookrightarrow X$ of the closed point $p \hookrightarrow X$, the morphism $\pi\colon C \longrightarrow \operatorname{Spec} R$ descends to a proper smooth morphism $C' \longrightarrow U$ with a closed fiber $C_0' = C' \times_U p \cong C_0$. By cohomological purity (see [11]) $H^q_{C_0'}(C',\mathbb{G}_m) \cong H^{q-2}(C_0,\mathbb{Q}/\mathbb{Z})$ for all $q \geq 3$. Taking the limit over all such $U \hookrightarrow X$ yields $H^q_{C_0}(C,\mathbb{G}_m) \cong H^{q-2}(C_0,\mathbb{Q}/\mathbb{Z})$ for all $q \geq 3$. Therefore $H^2(C_\eta,\mathbb{G}_m) \cong H^1(C_0,\mathbb{Q}/\mathbb{Z})$, $H^3(C_\eta,\mathbb{G}_m) \cong H^2(C_0,\mathbb{Q}/\mathbb{Z}) \cong \mu$ and $H^q(C_\eta,\mathbb{G}_m) \cong H^{q-1}(C_0,\mathbb{Q}/\mathbb{Z}) = 0$ for all $q \geq 4$. The isomorphism ${}_nH^2(C_\eta,\mathbb{G}_m) \cong H^1(C_0,\mathbb{Z}/n)$ is the Gysin map [10, p. 244]. Given any λ in $H^1(C,\mathbb{Z}/n)$ there is a corresponding λ_0 in $H^1(C_0,\mathbb{Z}/n)$. Let λ_η denote the image of λ in the group $H^1(C_\eta,\mathbb{Z}/n)$ under the natural map. Let λ_η also denote the corresponding cyclic Galois cover of C_η with group $\langle\sigma\rangle$. Using the cyclic Galois cover λ_η and the trivial factor set t we form a cyclic crossed product algebra $\Delta(\lambda) = (\lambda_\eta/C_\eta, \sigma, t)$ which represents a class in ${}_n\mathrm{B}(C_\eta)$. Consider the ramification divisor of $\Delta(\lambda)$ on C. Along the divisor C_0 of C, the ramification of $\Delta(\lambda)$ is the element λ_0 of the group $H^1(C_0, \mathbb{Z}/n)$. Therefore, the correspondence $\lambda \mapsto \Delta(\lambda)$ induces an isomorphism $H^1(C, \mathbb{Z}/n) \cong {}_n\mathrm{B}(C_\eta)$. Every Azumaya algebra over C_η whose Brauer class is annihilated by n is Brauer equivalent to a cyclic crossed product of the form $(\lambda_\eta/C_\eta, \sigma, t)$, hence is split by λ_η for some λ. ∎

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